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5. Exercise in Mathematics and Scientific Computing for Inverse Problems

(Fréchet derivative, surface potential)

DEFINITION 1. (Fréchet differential)
Suppose that $F : U \to Y$ and $u \in U$. We say that F is Fréchet differentiable *at u if there exists $A \in L(X, Y)$ such that*

$$F(u + h) - F(u) - Ah = o(\|h\|)$$

as $h \to 0$. The linear mapping A is uniquely determined and is called Fréchet differential *of F at u, and denoted as $F'(u)$. If F is differentiable at all $u \in U$ we say that F ist differentiable in U.*

1. Problem (3 points)

(chain rule)
Let $F : U \to Y$, $F(U) \subset V$, $G : V \to Z$ and H be the composite map $H = G \circ F$, so that $H(u) = G(F(u)),\ u \in U$. If F is differentiable in U and G is differentiable in V, then show that H is differentiable in U and

$$H'(u)h = G'(F(u))[F'(u)h]$$

2. Problem (3 points)

(product rule)
Let $F_1 : U \to Y$, $F_2 : U \to Z$ be Fréchet differentiable in U, and define $H : U \to W$ by $H(u) = B(F_1(u), F_2(u))$ with another Banach-space W and a bilinear mapping $B : Y \times Z \to W$ which is bounded. Then show that H is Fréchet differentiable in U and

$$H'(u)h = B(F'(u)h, G(u)) + B(F(u), G'(u)h)$$

DEFINITION 2. (harmonic)
A twice continuously differentiable real-valued function u, defined on a domain $D \subset \mathbb{R}^m$, $m = 2, 3$, is called harmonic *if it satifies* Laplace's equation

$$\Delta u = 0 \quad \textit{in } D$$

3. Problem

(4 points)

Show that for the harmonic function $\Phi(x,y) = \frac{1}{2\pi} \ln \frac{1}{|x-y|} \quad \in \mathbb{R}^2 \backslash \{y\}$ and a bounded domain $D \subset \mathbb{R}^2$ of class C^1 the following holds

$$\int_{\partial D} \frac{\partial \Phi(x,y)}{\partial \nu(y)} \,\mathrm{d}\, s(y) = \begin{cases} -1, & x \in D, \\ -\frac{1}{2}, & x \in \partial D, \\ 0, & x \in \mathbb{R}^m \backslash \bar{D}. \end{cases}$$

(Hint: use Green's Theorem)

4. Problem

(3 points)

Write a matlab function which calculates the integral from problem 3, so numericly show the analytical proof for a chosen domain (e.g. a circle).

Sum of Points: 13