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http://www.math.uni-goettingen.de/schulz/

7. Exercise in Mathematics and Scientific Computing for Inverse Problems

(weak convergence, Gâteaux derivative)

DEFINITION 1. (weak convergence)
A sequence (φ_n) *of elements from a Hilbert space* X *is called* weakly convergent *to an element* $\varphi \in X$ *if*

$$\lim_{n\to\infty} (\psi, \varphi_n) = (\psi, \varphi)$$

for all $\psi \in X$. *We will write* $\varphi_n \rightharpoonup \varphi, n \to \infty$

1. Problem (3 points)

Show that a weakly convergent sequence in a Hilbert space is bounded.
(Hint: use that by the uniform boundedness principle a pointwise convergent sequence of linear Operators is uniformly bounded).

2. Problem (4 points)

A compact linear operator $A : X \to Y$ (X, Y normed spaces) maps weakly convergent sequences to norm convergent sequences.
(Hint: a weakly convergent sequence in a Hilbert space is bounded. Every norm convergent sequence is also weakly convergent and their limits are the same)

DEFINITION 2. (Gâteaux differential)
Suppose that $F : U \to Y$ *and* $u \in U$. *We say that F is* Gâteaux differentiable *at* u *if there exists* $A \in L(X, Y)$ *such that*

$$\frac{1}{t}[F(u + th) - F(u)] = A(u)h$$

as $t \to 0$ *for all* $h \in X$ *with* $\|h\| = 1$. *The linear mapping* A *is uniquely determined and is called* Gâteaux differential *of* F *at* u, *and denoted as* $F'(u)$. *If* F *is differentiable at all* $u \in U$ *we say that* F *ist differentiable in* U.

3. Problem

(4 points)

Consider the bounded linear integral operator $A : C[0,1] \to C[0,1]$ defined as:

$$(A\varphi)(x) := \int_0^1 \frac{1}{\sqrt{x + r(x) - (y + r(y))}} \varphi(y) \, \mathrm{d}\, y$$

Show that A is Gâteaux differentiable in respect to r.

Sum of Points: 11